Simplicial and Cellular Structures in Algebraic Topology

Dai Tamaki

Abstract

This is a summary of my talk at Studio Phones seminar on February 22nd, 2012.

In this talk, I would like to explain major roles in homotopy theory played by simplicial and cellular structures. Homotopy theory is a theory of deformations. According to Gel′fand[1], therefore, homotopy theory is one of the most fundamental fields in mathematics. Homotopy theory is, however, usually regarded as a part of topology. What is topology? From my viewpoint, topology has the following three aspects:

1. treat continuous deformations seriously.
2. use invariants heavily.
3. provide global viewpoints.

From this point of view, the first and the third play important roles in homotopy theory. Let me explain how.

1 From Simplicial Complexes to Simplicial Sets and CW complexes

In an attempt to answer Heegaard's criticism on his ``Analysis Situs'' [Poi96], Poincaré decided to use triangulations of manifolds. The target manifold is cut into pieces by smooth ``surfaces''. Later people decided to abandon the smoothness requirement. Each piece is supposed to be homoeomorphic to an interior of a convex polytope.

According to Dieudonné's book on the history of algebraic and differential topology [Die89], it was Lefschetz [Lef08] who defined the concept which is now called Euclidean simplicial complex.

Definition 1.1. A *(Euclidean) polyhedral complex* in $\mathbb{R}^n$ is a subspace K of $\mathbb{R}^n$ equipped with a family of a finitely many number of maps $\{\varphi_i : P_i \longrightarrow K \mid i = 1, \cdots, n\}$ satisfying the following conditions:

1. Each P_i is a convex polytope;
2. Each φ_i is an affine equivalence onto its image;
3. $K = \bigcup_{i=1}^{n} \varphi_i(P_i)$;
4. For $i \neq j$, $\varphi_i(P_i) \cap \varphi_j(P_j)$ is a proper face of $\varphi_i(P_i)$ and $\varphi_j(P_j)$.

P_i's or $\varphi_i(P_i)$ are called *generating polytopes* .

When all generating polytopes are simplices, $(K, \{\varphi_i\})$ is called a *Euclidean simplicial complex*.

[1] See Introduction of the draft of the book by Kontsevich and Soibelman on deformation theory.

The notion of Euclidean simplicial (polyhedral) complexes evolved into three structures: abstract simplicial complex, ordered simplicial complex, and cell complex.

Abstract simplicial complexes are one of the fundamental objects of study in modern combinatorics.

Definition 1.2. An *abstract simplicial complex* on the vertex set S is a collection X of finite subsets of S,

$$X \subset 2^S,$$

satisfying the following condition: if $\sigma \in X$, $\tau \subset \sigma$, then $\tau \in X$. We usually assume X is *essential*, i.e.,

$$\bigcup_{\sigma \in X} \sigma = S.$$

An element in X is called a *face* or a *simplex* of X. When the cardinality of a simplex $\sigma \in X$ is $n+1$, we say σ is an n-dimensional simplex.

I do not know who introduced the current definition of abstract simplicial complexes. One of the key facts which make abstract simplicial complexes so useful in combinatorics can be illustrated by the following diagram:

$$\begin{array}{ccc} \mathbf{ASC} & \xrightarrow{F} & \mathbf{Posets} \\ {\scriptstyle \mathrm{Sd}}\downarrow & \swarrow{\scriptstyle \Delta} & \\ \mathbf{OSC}, & & \end{array} \tag{1}$$

where **ASC**, **OSC**, and **Posets** are categories of abstract simplicial complexes, of ordered simplicial complexes, and of posets, respectively. Functors F, Sd, and Δ are the face poset functor, the barycentric subdivision functor, and the order complex functor, respectively.

Definition 1.3. An *ordered (abstract) simplicial complex* is an abstract simplicial complex K whose vertex set S is partially ordered and the induced ordering on each simplex is a total order.

Definition 1.4. The set of all faces of an abstract simplicial complex K is denoted by $F(K)$. It is regarded as a poset under the inclusions and is called the *face poset* of K.

Definition 1.5. For a poset P, define an ordered simplicial complex $\Delta(P)$ with vertex set P by

$$\Delta(P) = \{\sigma \subset P \mid \sigma \text{ is totally ordered.}\}.$$

This is called the *order complex* of P.

Definition 1.6. For an abstract simplicial complex K, the ordered simplicial complex $\Delta(F(K))$ is called the *barycentric subdivision* and is denoted by $\mathrm{Sd}(K)$.

It was Henry Whitehead [Whi49] who introduced the closure finiteness and weak topology conditions on cell complexes and defined CW complexes, which generalize Euclidean polyhedral complexes.

Definition 1.7. An *n-cell* in a topological space X is a subset $e \subset X$ equipped with a continuous map $\varphi : D^n \to \overline{e}$ whose restriction $\varphi|_{\mathrm{Int}(D^n)}$ is a homeomorphism onto e.

A *cell complex* is a topological space X together with a family of cells $\{\varphi_\lambda : D^{n_\lambda} \to e_\lambda\}_{\lambda \in \Lambda}$ with the following properties:

1. $X = \bigcup_{\lambda \in \Lambda} e_\lambda$.

2. For an n-cell e_λ, $\partial e_\lambda = \overline{e_\lambda} - e_\lambda$ is a union of cells of dimension $\leq n-1$.

A cell complex X is called a *CW complex* if it satisfies the following two conditions:

1. it is closure finite i.e. $\overline{e_\lambda}$ is covered by a finite number of cells for each λ, and
2. it has the weak topology with respect to the covering $\{\overline{e_\lambda}\}_{\lambda\in\Lambda}$.

The triangle (1) does not generalize to CW complexes. We obtain, however, the following diagram by restricting our attention to regular CW complexes:

$$\begin{array}{ccc} \mathbf{CW}_{\text{reg}} & \xrightarrow{F} & \mathbf{Posets} \\ {\scriptstyle \mathrm{Sd}}\downarrow & & \downarrow{\scriptstyle \Delta} \\ \mathbf{CW}_{\text{reg}} & \xleftarrow[|-|,]{} & \mathbf{OSC} \end{array}$$

where $\mathbf{CW}_{\text{reg}}$ is the category of regular CW complexes and $|-|$ is the geometric realization functor.

Definition 1.8. A cell complex X is said to be *regular* if $\varphi_\lambda : D^{n_\lambda} \to \overline{e_\lambda}$ is a homeomorphism.

Definition 1.9. For an ordered simplicial complex K, define a topological space $|K|$ by

$$|K| = \left(\coprod_n K_n \times \Delta^n\right)\Big/_{\sim},$$

where Δ^n is the standard n-simplex and $\sim$ is the equivalence relation generated by

$$(d_i(x), s) \sim (x, d^i(s)),$$

where d_i is the operation which removes the i-th vertex[2] and $d^i : \Delta^{n-1} \hookrightarrow \Delta^n$ is the inclusion onto the face opposite to the i-th vertex. The space $|K|$ is called the *geometric realization* of K.

Definition 1.10. For a regular cell complex X, the cell complex $|\Delta(F(X))|$ is called the *barycentric subdivision* of X and is denoted by $\mathrm{Sd}(X)$.

Another importance of the concept of ordered simplicial complex is that it was the origin of simplicial sets introduced by Kan [Kan57] under the name ``semi-simplicial complexes".

Definition 1.11. A *simplicial set* X consists of a sequence of sets

$$X_0, X_1, \dots$$

and maps

$$\begin{aligned} d_i &: X_{n+1} \longrightarrow X_n \ (0 \leq i \leq n+1) \\ s_i &: X_{n-1} \longrightarrow X_n \ (0 \leq i \leq n-1) \end{aligned}$$

satisfying the following relations:

$$\begin{cases} d_i \circ d_j = d_{j-1} \circ d_i, & i < j \\ d_i \circ s_j = s_{j-1} \circ d_i, & i < j \\ d_j \circ s_j = 1 = d_{j+1} \circ s_j & \\ d_i \circ s_j = s_j \circ d_{i-1}, & i > j+1 \\ s_i \circ s_j = s_{j+1} \circ s_i, & i \leq j. \end{cases}$$

[2]Note that since X is ordered the vertices of an n-dimensional simplex is numbered from 0 to n.

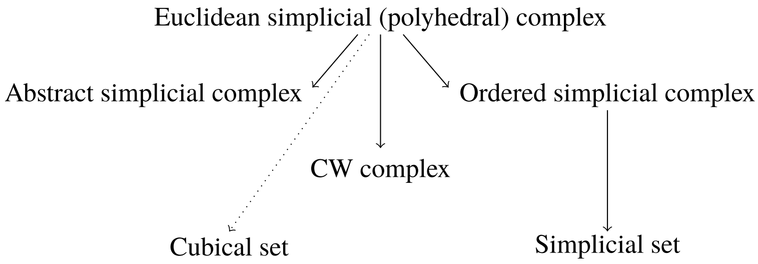
Before this work, Kan also investigated a possibility of using cubical complexes in [Kan55]. Simplicial sets, however, have the following simple characterization.

Definition 1.12. The category of isomorphism classes of finite totally ordered sets and order preserving maps is denoted by Δ.

Proposition 1.13. *The category of simplicial sets is isomorphic to category of functors* $\mathrm{Funct}(\Delta^{\mathrm{op}}, \mathbf{Sets})$. *In other words, the category of simplicial sets is the category of presheaves on* Δ *with values in* **Sets**.

Definition 1.14. The category of simplicial sets is denoted by $\mathbf{Sets}^{\Delta^{\mathrm{op}}}$.

We have seen the following developement of simplicial and cellular structures so far:



2 Simplicial and Cellular Structures in Homotopy Theory

It was Kan who first tried to develop general ``homotopy theory'' based on these ``well-behaved spaces''. Kan first tried to develop homotopy theory based on cubical complexes in a series of papers [Kan55; Kan56a; Kan56b; Kan56c]. Then developed an abstract homotopy theory based on simplicial sets in [Kan57].

The concept of ``abstract homotopy theory'' was made explicit by Quillen in [Qui67] as model structures. The following simplified definition can be found in a paper [JT07] by Joyal and Tierney.

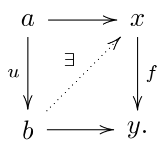
Definition 2.1. Let $\boldsymbol{C}$ be a category closed under finite limits and colimits. A *model structure* on $\boldsymbol{C}$ is a triple (C, W, F) of subcategories satisfying the following conditions:

1. W satisfies the ``two-out-of-three'' property.
2. The pair $(C \cap W, F)$ is a weak factorization system.
3. The pair $(C, W \cap F)$ is a weak factorization system.

A category with a model structure is called a *model category*. Morphisms in $C \cap W$ and $W \cap F$ are called *acyclic cofibrations* and *acyclic fibrations*, respectively.

Weak factorization systems are characterized by lifting properties.

Definition 2.2. Let $\boldsymbol{C}$ be a category and consider a diagram



We say f has the *right lifting property* with respect to u if, for any commutative square as above, there exists a morphism $b \to x$ filling in the diagram. In this case we also say that u has the *left lifting property* with respect to f.

Definition 2.3. Let $\boldsymbol{C}$ be a category. A *weak factorization system* is a pair of subcategories (A, B) satisfying the following conditions:

1. Any morphism f in $\boldsymbol{C}$ can be factored as $f = p \circ i$ with i in A and p in B, i.e. $\boldsymbol{A}$ has (A, B)-factorizations.

2. A morphism f has the right lifting property with respect to every morphism in A if and only if f belongs to B.

3. A morphism g has the left lifting property with respect to every morphism in B if and only if f belongs to A.

Remark 2.4. Hovey requires that the weak factorizations in his definition of model category to be functorial. There are, however, examples of model categories whose weak factorizations are not functorial. See Chorny's paper [Cho03].

One of the most important discovery of Henry Whitehead is that CW complexes behave extremely well with respect to homotopy. For example, any pair (X, A) of a CW complex X and its subcomplex A has the homotopy extension property (HEP). On the other hand, covering homotopy property of fiber bundles evolved into the definition of fibrations by Hurewicz [Hur55] and Serre [Ser51]. Quillen's definition of model structure is based on these structures.

Quillen constructed model structures on the category $\mathbf{Sets}^{\Delta^{\mathrm{op}}}$ of simplicial sets and the category **Spaces** of topological spaces. He invented so-called the small object argument for this purpose. The point is that the model structures on these categories are generated by ``cellular structures". We use transfinite compositions to define ``cell complexes" based on a class of morphisms I.

Definition 2.5. Let $\boldsymbol{C}$ be a category and I be a small subcategory of $\boldsymbol{C}$ closed under colimits. Define the class I-cell of *relative I-cell complexes* in $\boldsymbol{C}$ by a transfinite induction as follows: Morphisms in I belong to I-cell. A morphism $f : A \to B$ is in I-cell if there exists an ordinal λ and a colimit-preseving functor

$$X : \lambda \longrightarrow \boldsymbol{C}$$

satisfying the following properties:

1. f is the composition of X.

2. For each β with $\beta + 1 < \lambda$, there is a pushout square

$$\begin{array}{ccc} C_\beta & \longrightarrow & X_\beta \\ {\scriptstyle g_\beta}\downarrow & & \downarrow \\ D_\beta & \longrightarrow & X_{\beta+1} \end{array}$$

   such that $g_\beta \in I_1$.

We say an object A in $\boldsymbol{C}$ is an *I-cell complex* if the canonical morphism

$$\emptyset \longrightarrow A$$

belongs to I-cell, where $\emptyset$ is the initial object in $\boldsymbol{C}$.

Definition 2.6. Let $\boldsymbol{C}$ be a category closed under small colimits and κ be a cardinal.

1. An object x is *κ-small relative to a subcategory $\boldsymbol{D}$* if for every regular cardinal $\lambda \geq \kappa$ and every λ-sequence

$$a_0 \to a_1 \to a_2 \to \cdots \to a_\beta \to \cdots$$

with $\beta < \lambda$ in $\boldsymbol{C}$ such that $a_\beta \to a_{\beta+1}$ is in $\boldsymbol{D}$ for every cardinal β satisfying $\beta + 1 < \lambda$, the map of sets

$$\operatorname*{colim}_{\beta<\lambda} \boldsymbol{C}(x, a_\beta) \longrightarrow \boldsymbol{C}\left(x, \operatorname*{colim}_{\beta<\lambda} a_\beta\right)$$

is an isomorphism.

2. An object x is *κ-small relative to a class of morphisms I* if it is κ-small relative to the subcategory of relative I-cell complexes.

An object x is said to be *small* relative to I if x is κ-small relative to I for some cardinal κ.

There is a canonical way of producing a weak factorization system from such a class of morphisms or a subcategory I.

Theorem 2.7 (The Small Object Argument)**.** *Let $\boldsymbol{C}$ be a category closed under arbitrary small colimits and I be a small subcategory. Suppose the domains of morphisms in I are small relative to I. Then there exists a functorial weak factorization system (γ, δ) on $\boldsymbol{C}$ such that $\gamma(f)$ is in I-cell and $\delta(f)$ is in I-inj for any morphism f in $\boldsymbol{C}$.*

The class I-inj used in the above theorem is defined as follows.

Definition 2.8. Let I be a class of morphisms (i.e. a subcategory) in a category $\boldsymbol{C}$. A morphism f is

1. *I-injective* if it has the right lifting property with respect to morphisms in I,
2. *I-projective* if it has the left lifting property with respect to morphisms in I,
3. *I-cofibration* if it has the left lifting property with respect to every I-injective morphism,
4. *I-fibration* if it has the right lifting property with respect to every I-projective morphism.

The subcategories of I-injectives, I-projectives, I-cofibrations, and I-fibrations are denoted by I-inj, I-proj, I-cof, and I-fib, respectively.

Definition 2.9. Let $\boldsymbol{C}$ be a category. We say a class of morphisms *I permits the small object argument* if the domains of elements in I are small relative to I.

Definition 2.10. A *cofibrantly generated model category* is a model category $\boldsymbol{C}$ equipped with two classes of morphisms I and J satisfying the following conditions:

1. I permits the small object argument.
2. A morphism is a acyclic fibration if and only if it has the right lifting property with respect to all elements in I.
3. J permits the small object argument.
4. A morphism is a fibration if and only if it has the right lifting property with respect to all elements of J.

Elements in I and J are called *generating cofibrations* and *generating acyclic cofibrations*, respectively.

Theorem 2.11. *Let $\boldsymbol{C}$ be a category closed under small limits and colimits and W be a subcategory that is closed under retracts and satisfies the two-out-of-three axiom.*

Suppose that I and J are classes of morphisms of $\boldsymbol{C}$ satisfying the following properties:

1. *I and J permit the small object argument.*
2. *$J-\mathrm{cof} \subset I-\mathrm{cof} \cap W$.*
3. *$I-\mathrm{inj} \subset J-\mathrm{inj} \cap W$.*
4. *One of the following conditions holds:*
   (a) *$I-\mathrm{cof} \cap W \subset J-\mathrm{cof}$,*
   (b) *$J-\mathrm{inj} \cap W \subset I-\mathrm{inj}$.*

Then there exists a cofibrantly generated model structure on $\boldsymbol{C}$ in which W is the subcategory of weak equivalences, I is the class of generating cofibrations, and J is the class of generating acyclic cofibrations.

In a cofibrantly generated model category, ``cells" are generators of cofibrations. In particular, in the Quillen model structure of topological spaces, cofibrations are retracts of relative CW complexes.

Quillen also proved that, under his model structures, the geometric realization functor $|-|$ and the singular simplicial set functor S define a Quillen equivalence

$$|-| : \mathbf{Sets}^{\Delta^{\mathrm{op}}} \longleftrightarrow \mathbf{Spaces} : S.$$

Definition 2.12. Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be model categories.

- We say a functor $f : \boldsymbol{C} \to \boldsymbol{D}$ is a *left Quillen functor* if f is a left adjoint and preserves cofibrations and trivial cofibrations.
- Dually, we say a functor $g : \boldsymbol{D} \to \boldsymbol{C}$ is a *right Quillen functor* if g is a right adjoint and preserves fibrations and trivial fibrations.
- A *Quillen adjunction* is an adjunction

  $$\varphi : \boldsymbol{D}(f(x), y) \xrightarrow{\cong} \boldsymbol{C}(x, g(y))$$

  such that f is a left Quillen functor. We denote a Quillen adjunction by

  $$(f, g, \varphi) : \boldsymbol{C} \longrightarrow \boldsymbol{D}.$$

- We say a Quillen adjunction

  $$(f, g, \varphi) : \boldsymbol{C} \longrightarrow \boldsymbol{D}$$

  is a *Quillen equivalence* if, for all cofibrant x in $\boldsymbol{C}$ and fibrant y in $\boldsymbol{D}$, we have

  $$W_{\boldsymbol{D}} \cap \boldsymbol{D}(f(x), y) = \varphi^{-1}(W_{\boldsymbol{C}} \cap \boldsymbol{C}(x, g(y))).$$

Theorem 2.13. *The geometric realization and the singular simplicial set functors define a Quillen equivalence between* $\mathbf{Sets}^{\Delta^{\mathrm{op}}}$ *and* $\mathbf{Spaces}$.

In particular, the geometric realization $|X|$ of any simplicial set X has a canonical structure of CW complex and we have a weak homotopy equivalence

$$|S(Y)| \xrightarrow[w]{\simeq} Y$$

for any topological space Y. In other words, any topological space can be replaced by a CW complex up to a weak homotopy equivalence.

After Quillen's discovery, model categories have been used in many fields. We can refine classical homological algebra based on triangulated categories by using stable model structures.

Why is the concept of model categories so useful?

One of the answers to this question is the following.

Fact 2.14. *Triangulated categories suffer the well-known defect that the cone construcion and other ``colimit-type'' constructions are not functorial. By lifting to model categories, we may define homotopy colimits of any diagrams under a mild condition.*

One of the conditions on a model category under which we may perform homotopy limits and colimits is the existence of a simplicial enrichment. This is one of main topics in Hirschhorn's book [Hir03]. It should be also noted that triangulated categories are often constructed as a homotopy category of a category with richer structures such as (stable) dg categories, A_∞-categories, and model categories.

Dwyer and Kan [DK80c; DK80a; DK80b] found that simplicial structures can be used to extend the construction of the homotopy category of a model category to categories with a designated class of ``weak equivalences''.

Theorem 2.15. *Given a category $\boldsymbol{C}$ and a subcategory W with $\boldsymbol{C}_0 = W_0$, there exists a simplicial category $L_W\boldsymbol{C}$ with*

$$\pi_0(L_W\boldsymbol{C}(x,y)) \cong \boldsymbol{C}[W^{-1}](x,y).$$

Recall that, for a model category $\boldsymbol{C}$, we define the homotopy category $\mathrm{Ho}(\boldsymbol{C})$ by formaly inverting the class W of weak equivalences, i.e. $\mathrm{Ho}(\boldsymbol{C}) = \boldsymbol{C}[W^{-1}]$. This naive definition, however, suffers from a set theoretical difficulty. The discovery of Dwyer and Kan suggests the usefulness of simplicial structure in localizations of categories.

An interesting and important fact is that the existence of simplicial enrichment also leads to another approach to homotopy categories, i.e. the theory of $(\infty,1)$-categories.

3 Simplicial and Cellular Structures in Higher Category Theory

Definition 3.1. A *simplicial category* is a category enriched over the category $\mathbf{Sets}^{\Delta^{\mathrm{op}}}$. The category of simplicial categories is denoted by $\mathbf{Sets}^{\Delta^{\mathrm{op}}}$**-Cats**.

The work of Dwyer and Kan suggests that the category $\mathbf{Sets}^{\Delta^{\mathrm{op}}}$**-Cats** can be regarded as the category of categories whose homotopy categories can be defined. It turns out there are several other candidates for such models for ``homotopy theory of homotopy theories''.

One of them also originates in an old work of Kan. Recall that an object x in a model category $\boldsymbol{C}$ is said to be *fibrant* if the canonical morphism $x \to *$ to the terminal object is a fibration.

Definition 3.2. Fibrant objects in $\mathbf{Sets}^{\Delta^{\mathrm{op}}}$ are called *Kan complexes*.

Proposition 3.3. *A simplicial set X is a Kan complex if and only if, for $n \geq 0$, $0 \leq i \leq n$, and any morphism of simplicial sets*

$$\varphi : \Lambda_i^n \longrightarrow X,$$

there exists an extension $\tilde{\varphi} : \Delta^n \to X$, where Λ_i^n is the simplicial subset of Δ^n generated by all faces but the i-th one.

The nerve of a small category has an analogous property.

Proposition 3.4. *For a simplicial set K, the following conditions are equivalent:*

1. *There exists a small category X with $K \cong N(X)$.*
2. *For any $n \geq 0$, $0 < i < n$ and a morphism $\varphi : \Lambda_i \to K$, there exists a unique extension $\tilde{\varphi} : \Delta^n \to K$ of φ.*

As a class containing these two classes of simplicial sets, Boardman and Vogt [BV73] introduced and studied simplicial sets satisfying the *restricted Kan condition*, called *weak Kan complexes* or *quasicategories* these days.

Definition 3.5. A *quasicategory* is a simplicial set K satisfying the condition that, for any $n \geq 0$, $0 < i < n$ and a morphism $\varphi : \Lambda_i \to K$, there exists an extension $\tilde{\varphi} : \Delta^n \to K$ of φ.

It was Joyal [Joy02] who first formulated the theory of quasicategories. Then Lurie [Lur09a] used it as a model for higher categories, more precisely categories whose higher morphisms are invertible.

It is well-known that the nerve functor

$$N : \mathbf{Cats} \longrightarrow \mathbf{Sets}^{\Delta^{\mathrm{op}}}$$

is an embedding. By Proposition 3.4, the nerve functor embeds **Cats** into the category **QCats** of quaiscategories. It is also well-known that the nerve functor has a left adjoint

$$\tau_1 : \mathbf{Sets}^{\Delta^{\mathrm{op}}} \longrightarrow \mathbf{Cats},$$

which can be used to define the homotopy category construction for quasicategories.

Definition 3.6. The restriction of τ_1 to **QCats** is denoted by

$$h : \mathbf{QCats} \longrightarrow \mathbf{Cats}.$$

For a quaiscategory X, $h(X)$ is called the *homotopy category* of X.

Thus the category **QCats** of quasicategories is another model for the category of categories whose homotopy categories can be defined. Lurie found quasicategories are useful for developing derived algebraic geometry and other ``higher geometries and algebras" [Lurc; Lurb; Lura; Lurd; Lure; Lur09b].

It turns out there are many more models for homotopy theories of homotopy theories other than simplicial categories and quasicategories:

- Segal categories. [HS]
- complete Segal spaces. [Rez01]
- relative categories. [BK]

It is known that all these theories are equivalent to each other [Ber10]. And simplicial structures play essential roles in all of these theories.

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